MARINA’S FISH SHOP

A mathematically- and technologically-rich lesson

Roger Wander
The University of Melbourne
<rdwander@unimelb.edu.au>
Robyn Pierce
The University of Melbourne
<rupierce@unimelb.edu.au>

Introduction

In early 2008 researchers from the University of Melbourne’s *New Technologies for Teaching Mathematics* project created a lesson for the Year 10 students at their Victorian research schools.

Two important goals of secondary school mathematics education are to build students’ conceptual knowledge and to teach students to think mathematically. For decades it has been accepted that conceptual knowledge is characterised by rich relationships and that part of being able to think mathematically involves a disposition to explore mathematical ideas from a number of perspectives. The ease of demonstrating relationships between multiple representations of functions has repeatedly been put forward as one of the strengths of teaching using various mathematical software packages. The NCTM Standards document (1989) summarises the importance of designing tasks that use multiple representations:

> Different representations of problems serve as different lenses through which students interpret the problems and the solutions. If students are to become mathematically powerful, they must be flexible enough to approach situations in a variety of ways and recognize the relationships among different points of view. (p. 84)

Mathematics education research (see for example Kaput, 1992) supports the notion that seeing concepts represented in multiple ways both supports students’ sense-making and enriches their learning. However the experi-

ence of working with multiple representations is not the same for experts and novices. The design of lessons must be carefully considered in order not to create excessive cognitive load for the learner.

The study by the University of Melbourne team focused on one 'lesson', designed to provide exemplars for lessons where new powerful, mathematically able integrated documents (Nspire) are used to support students' exploration of various representations of a given mathematical problem.

Based on applications of quadratic functions, "Marina's Fish Shop" was designed as a capstone lesson utilising TI-Nspire CAS computer and handheld technology in exploring multiple representations of a mathematical problem set in a real world context. What follows is a description of the problem setting, then the six carefully guided student activities that formed the final product of that process.

1. Observing variation
2. Calculating total area
3. Graphing the area function from data
4. Finding the minimum area from the graph
5. Finding the minimum area exactly
6. Challenge: Producing a general solution

Initially students are presented with a non-specific question to encourage them to think broadly about the problem.

The problem setting

> Marina owns a fish shop, and wants to create a new sign above the shop. She likes geometric ideas, and thinks a square with a triangle looks like a fish. Marina draws a square with a horizontal diagonal, starting from the left wall of her shop. This makes the body. Then she extends two sides of the square as far as the right wall of the shop. This makes the tail. The shop is 10 metres wide. Marina soon realises that there is more than one possible configuration (see above) and wonders, "What is the best possible sign?" She uses mathematics to investigate.
>
> Marina is aware that certain aspects of both the body and the tail of the fish seem to change with each configuration, and seeks to develop some methods for recording and analysing these changes.

Students use both written and technology-assisted mathematics in Activities 1–3 to explore the patterns of data created by these configurations. Next the problem becomes specific as they are presented with a key constraint.

> At night, Marina's sign will show the interior of the fish design lit up against a black background. She is both an environmentalist and a smart business person, and wishes to use as little area of lighting as possible, to save money and energy.

In Activities 4 and 5, students find the measurements that Marina's sign should have to ensure this happens. In Activity 6 the results are generalised.

Lesson documentation and logistics

The overall lesson plan and the student worksheet are both multi-page Word documents. In addition there are similar but distinct student and teacher versions of the Nspire files. The former is formatted for use with a handheld device, whilst the latter is, ideally, run through a computer and data projector.

A good working knowledge of quadratic functions is assumed. Depending on students' familiarity with the technology as well as their ability to work independently, a time frame of 100 to 120 minutes is ideal for this lesson. Extension activities are provided for homework or assessment opportunities.

Activity 1: Observing variation

As the story of Marina's sign is told, a "flexible fish diagram" is manipulated by the teacher on the projection screen and then by students on their handheld devices to develop a sense of if and how the individual and combined areas are changing (Figure 1). Students are asked to estimate the length of the body of the fish (denoted by b) which will result in the maximum and minimum combined areas. They then answer some worksheet questions based on those estimates. We found that some students initially thought that the total area is a constant and, while most had a sense that change is occurring, there was incentive for further exploration to determine exactly what is going on. A practical suggestion to teachers arising out of the trials is to keep this and other activities moving, and to encourage very general estimates at this stage.

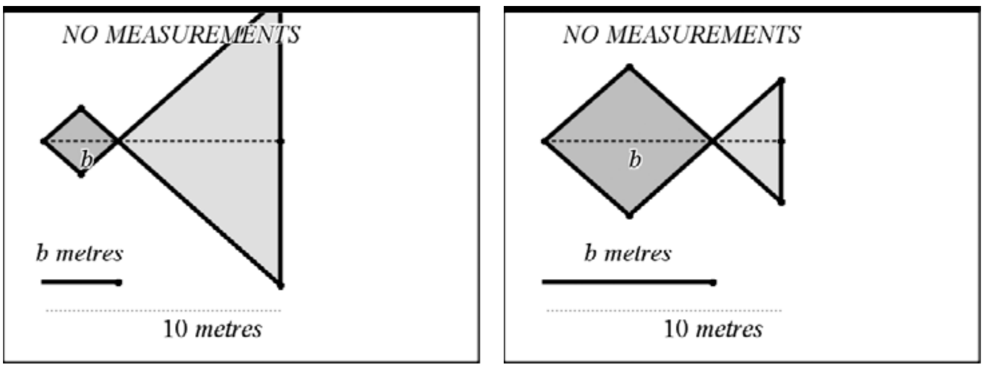


Figure 1. Nspire diagrams with b ≈ 4 and b ≈ 7.

Activity 2: Calculating total area

In this activity, students are presented with three diagrams on their worksheets, each with a slightly different configuration of body and tail. The first diagram, labelled "A small-bodied fish", has all critical dimensions clearly labelled, with $b = 4$; therefore the tail length = 6, and the sloping boundaries are presented in an approximate form correct to one decimal place. Simple area formulae are used to establish the total area of the fish.

The second diagram, labelled "A large-bodied fish", has $b = 8$ as its only given dimension; students are required to use Pythagoras' theorem and geometry facts to find all other dimensions; once again, the sloping boundary lengths are calculated approximately. Students are advised they may use the Calculator application of Nspire to perform the arithmetic and any algebraic manipulations they choose, as seen in Figure 2.

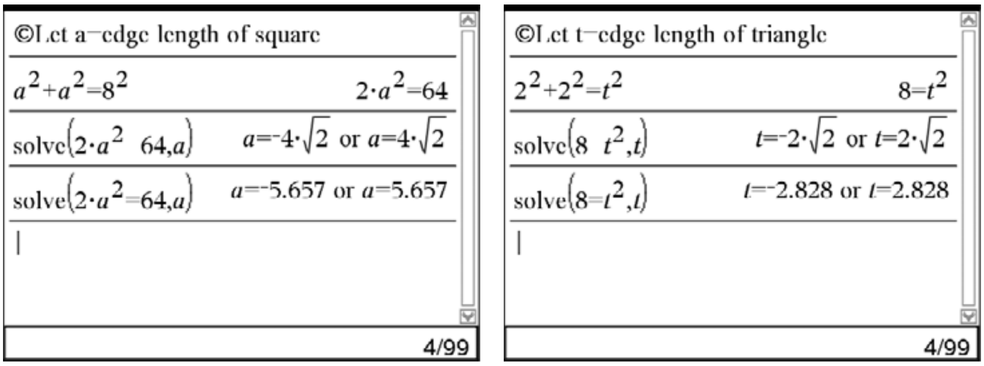


Figure 2. Sampling of possible Nspire-assisted calculations in Activity 2.

The third diagram is labelled "Any 10 m fish". Students are required to find an algebraic expression for the total area in terms of *b* using either handwritten or CAS-enabled algebra. Teachers trialling this lesson guided their students carefully through this process in order to finish the lesson in the allocated time. However revisiting the Activity 1 discussions regarding the issue of variability of the total area was instructive.

Activity 3: Graphing the area function from data

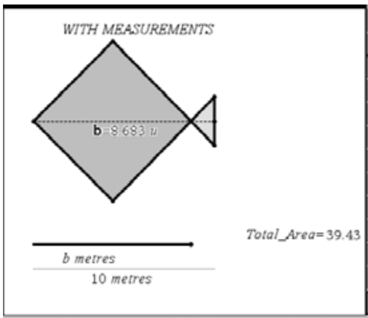
Students then move on to defining the area function, noting its quadratic nature. A table of values (Table 1) is completed using the function Total Area = $\frac{3}{2}b^2 - 20b + 100$. Students compare total area values for $b = 4$ and $b = 8$ to those found in Activity 1 where values of 43.84 and 36.41, respectively, may have been calculated earlier had surds not been employed.

Table 1. Table of values.

b m	Total area m^2	Coordinates (*b*, total area)
4	44	
8	36	
1.5		

The three points are then plotted on a grid on the worksheet, with students asked to connect them with an appropriate graph and explain what the graph says about the area of the fish. Students in the project schools were mixed in their ability to apply their knowledge of quadratic functions to the plotted points, but with prompting could see opportunities to refine their earlier estimates of *b*-values for minimum total area. Discussions of domain restrictions for *b* are useful here for refining maximum area estimates.

Nspire software is then employed to capture measurement data from another flexible fish diagram in a spreadsheet. This spreadsheet data is then linked to a scatterplot (Figure 3) which allows students to view an alternative representation of the possible fish-area based on body length, and also to judge the accuracy of their hand-drawn parabolas. The triple-paned screen is useful for teachers in their presentation of the lesson using a computer and data projector or interactive whiteboard.



	A bodyle...	B total_a...	C	D	E
♦	=capture('1	=capture('			
1	7.665	34.83			
2	5.942	34.12			
3	4.452	40.69			
4	2.865	55.01			
5	1.135	79.24			
6	9.356	44.18			
7	9.644	46.63			
8	0.173	96.58			
9	6.471	33.39			
10	8.683	39.43			

A1 =7.664793038287

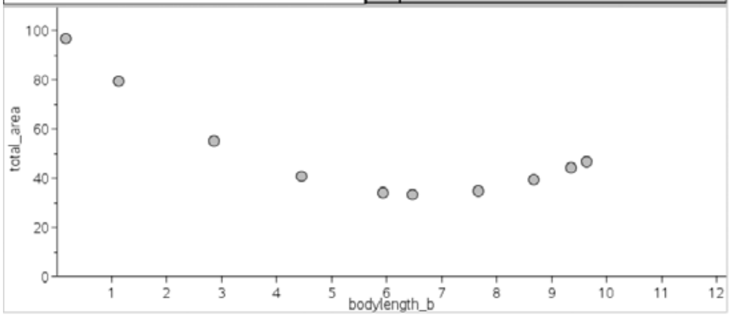


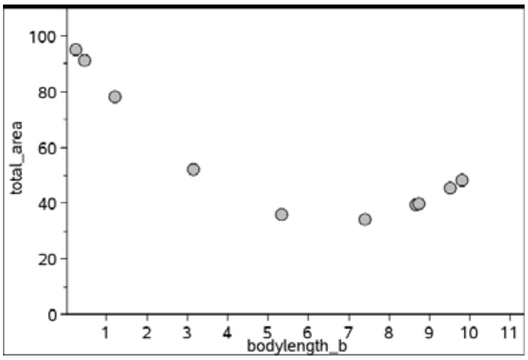
Figure 3. Teacher version for demonstration.

However, the handheld version of the file did not support this format effectively as it was too crowded. Thus, the separate screens are used in the student Nspire file. The regression graph and its equation are displayed for confirmation of the algebraic model (Figure 4).

The regression equation, with its use of the traditional x and y instead of the previously defined variables, requires careful linking to the variables named on the axes.

	A bodylength_b	B total_area	C	D
♦	=capture('b,0)	=capture(asum,0)		
1	8.661	39.3		
2	7.387	34.11		
3	5.325	36.04		
4	3.142	51.97		
5	1.2	78.16		
6	0.4564	91.18		
7	0.2406	95.27		
8	9.497	45.35		
9	8.73	39.72		
10	9.809	48.14		

C1



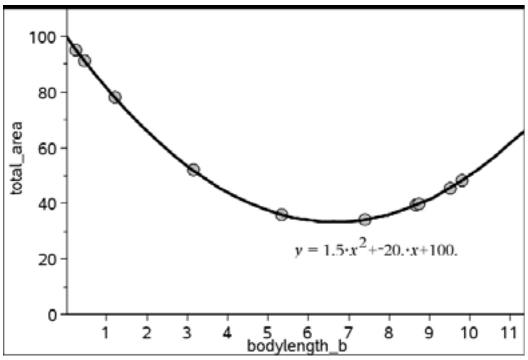


Figure 4. Student version with individual screens.

Activity 4: Finding the minimum area from the graph

Students now enter the function in an format in the Graphs & Geometry application, and using the “point-on” command find the location of the turning point, which is in approximate form in this application (Figure 5). Teachers may use the opportunity to steer students towards the hypothesis that two-thirds of the 10 m sign width is taken up by the body and one-third for the tail; also that the minimum total area is one third of the square of the sign width. It should also be noted that the domain of b as (0,10) is purposely not adhered to in the graph shown in Figure 6, allowing for students to name the maximum value of the total area by substitution into the function in the Calculator application. These facts are useful in the generalisation process seen at the end of the lesson. The activity concludes with the students drawing the resulting optimum configuration to scale on their worksheet.

Activity 5: Finding the minimum area exactly

With the previous method producing an approximate solution, students now use the symmetry of the parabola to find exact values for both b and the minimum area (Figure 7). With the minimum value around

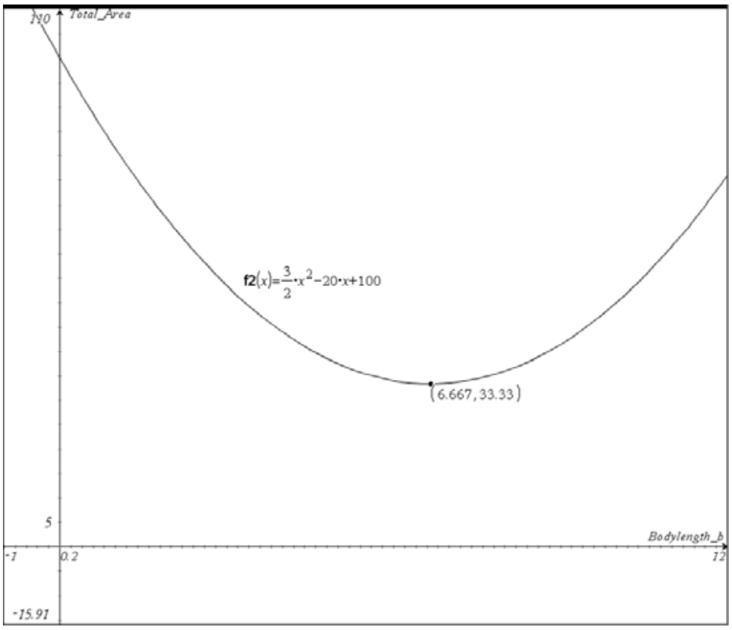


Figure 5. Graphical solution for minimum total area.

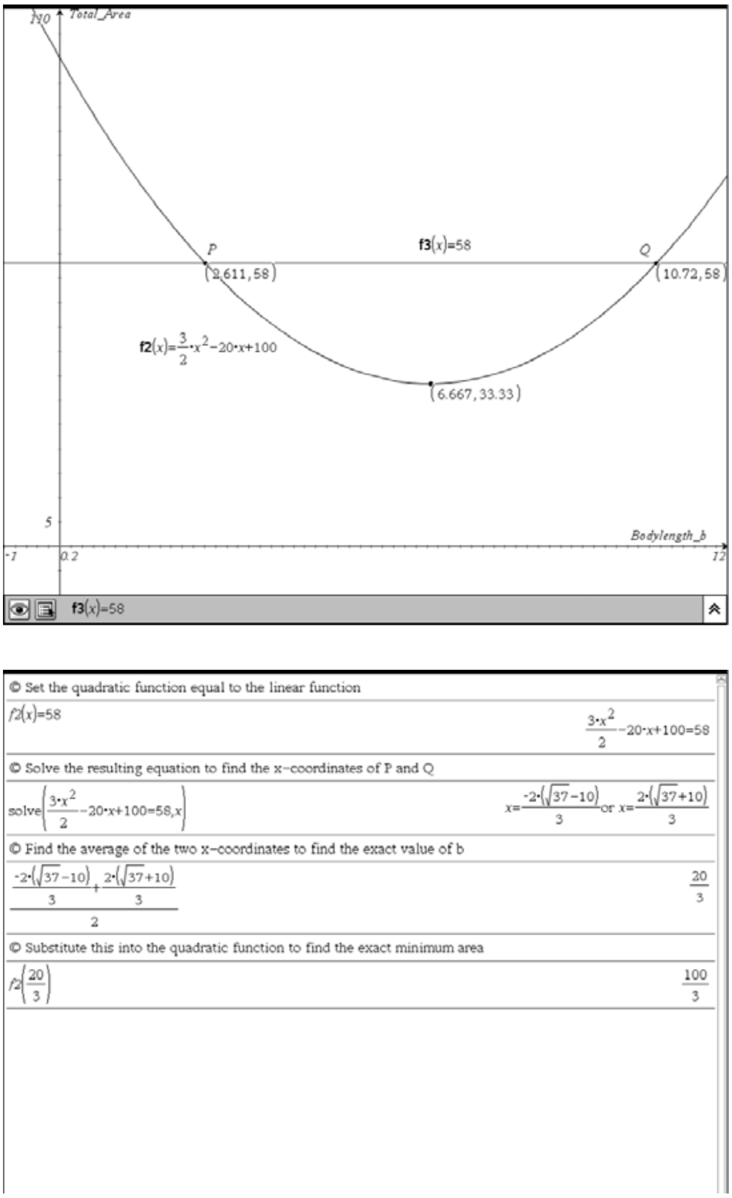


Figure 6. c = 58 in this example.

33, a horizontal linear function in the form $f(x) = c$ is sketched for any random value of c from say, 40 to 90. Intersection points with the parabola are named P and Q, and the students use the Calculator application to find the exact x-coordinates. Here, teachers will note the presence of $\frac{10}{3}$ in both values regardless of the c-value chosen. The mean of these becomes the exact value of b, and substitution into the area function completes the process.

Activity 6: Challenge: Producing a general solution

Two approaches are used for generalisation purposes. In the first of these, students are asked to produce an area function where the width of the sign becomes 14 m. They produce the function

Total Area = $\frac{3}{2}b^2 - 28b + 196$

which bears structural similarities to the original, prompting generalisation for any fish sign of width W m.

The second approach starts off assuming the sign width is W m; as the students are not expected to have a calculus background, this new function Total Area = $\frac{3}{2}b^2 - 2Wb + W^2$ cannot be sketched on the Graphs & Geometry screen. However, simple quadratic analysis will show this parabola to be similar to that used previously, and the y-intercept to be W^2. Students then produce a by-hand sketch of the general quadratic as well as the line $y = W^2$ (where P is the y-intercept). Using CAS-

enabled algebraic methods seen before in Activity 5 (Figure 7), the x-coordinate of Q is found and thus the exact values of b ($\frac{2w}{3}$) and the minimum total area ($\frac{w^2}{3}$) are found in terms of W. Comparison with Activity 5 results is used to establish the validity of these expressions.

Completing this activity within class was difficult to do in the allocated time. Depending on the skills of the group, it might be set as a task to complete at home before the next class meeting.

© Set the quadratic function equal to the linear function

$\frac{3}{2}\cdot b1^2-2\cdot w\cdot b1+w^2=w^2$ $\qquad w^2-2\cdot b1\cdot w+\frac{3\cdot b1^2}{2}=w^2$

© Solve the resulting equation to find the x−coordinates of P and Q

$\text{solve}\left(w^2-2\cdot b1\cdot w+\frac{3\cdot b1^2}{2}=w^2, b1\right)$ $\qquad b1=\frac{4\cdot w}{3} \text{ or } b1=0$

© We knew b1=0 from y−intercept P, and exact value of b is easily found

$\frac{\frac{4\cdot w}{3}}{2}$ $\qquad \frac{2\cdot w}{3}$

© Substitute this into the quadratic function to find the exact minimum area

$\frac{3}{2}\cdot b1^2-2\cdot w\cdot b1+w^2|b1=\frac{2\cdot w}{3}$ $\qquad \frac{w^2}{3}$

Figure 7. Nspire requires new variable b1 as b previously defined.

Extensions, variations and access

The lesson plan suggests possible extensions to this investigation. An examination of when the body and tail areas are equal would be appropriate if Marina felt that a "well-balanced" sign would bring more customers to her shop. Another activity could focus on the individual and total perimeters if only the outline of the fish shape were to be lit.

The lesson as a whole is long, with many parts. As a lesson at the culmination of Year 10 work on quadratics we expected students to move quickly through these activities. Teachers might choose to use only one or more sections of the lesson either during or at the end of the unit.

Access to the files for Marina's Fish Shop is available on the RITEMATHS website at http://extranet.edfac.unimelb.edu.au/DSME/RITEMATHS.

Acknowledgements

The authors wish to acknowledge the major contributions of Kaye Stacey and Lynda Ball from the University of Melbourne to the design of this lesson. We also thank Texas Instruments and the teachers and students from the New Technologies for Teaching Mathematics project schools 2008.

References

Kaput, J. (1992). Technology and mathematics education. In D. Grouws (Ed.), A handbook of research on mathematics teaching and learning (pp. 515-556). New York: MacMillan.

National Council of Teachers of Mathematics (1989). Curriculum and Evaluation Standards for School Mathematics. Reston VA: NCTM.